EXACT SOLITONS IN THE NONLOCAL GORDON EQUATION

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ABSTRACT. We find exact monotonic solitons in the nonlocal Gordon equation $u_{tt} = J * u - u - f(u)$, in the case $J(x) = \frac{1}{2}e^{-|x|}$. To this end we come up with an inverse method, which gives a representation of the set of nonlinearities admitting such solutions. We also study $u^{(iv)} + \lambda u'' - \sin u = 0$, which arises from the above when we write it in traveling wave coordinates and pass to a certain limit. For this equation we find an exact 4π-kink and show the non-existence of 2π-kinks, using the analytic continuation method of Amick and McLeod.

1. INTRODUCTION

The sine-Gordon equation

$$u_{tt} = u_{xx} - \sin u \tag{1.1}$$

is one of the most popular PDEs. It was studied in connection with pseudo-spherical surfaces by Bäcklund in the 19th century [17], it is completely integrable and appears in many physical models. For example, the solitons solutions $u(x,t) = U(x - ct)$, $c^2 < 1$, $U(\mp\infty) = 0$, $U(\pm\infty) = 2\pi$, which have the exact form

$$u(x,t) = 4 \arctan \exp\left(\pm\frac{x - ct}{\sqrt{1 - c^2}}\right), \tag{1.2}$$

obey relativistic dynamics and may be treated as particles in a field theory [12].

The nonlocal sine-Gordon equation

$$u_{tt} = J * u - u - \sin u, \tag{1.3}$$

where $J * u(x) \equiv \int_R J(x - y)u(y)dy$, $\int_R J(x)dx = 1$ and $J(-x) = J(x)$, arises e.g., in Josephson tunnel junctions made of high temperature superconductors [1, 5]. Its discrete counterpart

$$\ddot{u}_n = u_{n+1} - 2u_n + u_{n-1} - \sin u_n. \tag{1.4}$$

has been used to model dislocations in crystals, as it represents the motion of a chain of gravity physical pendulums coupled by linear springs.

The study of solitons of (1.3) and (1.4) is of immediate interest. So far, only exact and numerical solutions have been found, of different type than (1.2) [13]. In [1], it was discovered that $u(x - ct) = 4\pi \arctan(x - ct)$, $c^2 = 1$, is a solution of the

sine-Hilbert equation

$$u_{tt} = \frac{1}{\pi} p.v. \int_R \frac{u_x(y)}{y-x} dx - \sin u, \tag{1.5}$$

which arises as a limit of (1.3) with an appropriate J. This 4π-kink carries two magnetic flux quanta [1]. In [5] the authors studied the case with $J(x) = \frac{1}{2}e^{-|x|}$ and stated a Hypothesis 1 [5, p.406], that there are no 2π-kinks for $0 < c^2 < 1$. Instead, they numerically determined that solitons develop periodic oscillations around 0 and 2π at $\mp\infty$, which has been interpreted as a Cherenkov radiation phenomenon. We mention that homoclinic asymptotically periodic waves were first formally [14] and then rigorously [4] constructed for a fourth order KdV equation. Also in [5], the authors found numerical evidence for 4π- and 6π-kinks for some values of c^2.

In this note, we approach the problem of existence or nonexistence of monotonic solitons of (1.3) from a different point of view. We replace $\sin u$ with a general $f(u)$, as we think that this problem is not structurally stable with respect to the nonlinearity. Namely, we study traveling wave solutions $u(x - ct)$ to

$$u_{tt} = \frac{1}{\varepsilon^2}(J_\varepsilon * u - u) - f(u), \tag{1.6}$$

where $J_\varepsilon(x) \equiv \frac{1}{\varepsilon}J(\frac{x}{\varepsilon})$. It is natural to consider this scaling, as formally with $\varepsilon \to 0$ such a solution approaches a traveling wave of (1.1). We take $J(x) = \frac{1}{2}e^{-|x|}$ as in [5]. $u(y) = u(x - ct)$ then satisfies

$$\varepsilon^2 c^2 u^{(iv)} + (1 - c^2)u'' - f(u) + \varepsilon^2 f(u)'' = 0. \tag{1.7}$$

Let $u' > 0$, $W(s) = \int_{-a}^s f$ and $z(s) = \frac{1}{2}[u'(u^{-1}(s))]^2 + \frac{W(s)}{c^2}$. We show below that (1.7) is then equivalent to

$$W = c^2 z - \frac{z + \frac{1}{2}\varepsilon^2 c^2 z'^2}{(\frac{1}{c^2} + \varepsilon^2 z'')^2}. \tag{1.8}$$

Thus f admits a monotonic soliton only if W is in the image of the mapping (1.8). Moreover, from (1.8) we see that for every f there exists a close f_ε which admits a monotonic soliton (Theorem 2.1 below). Likely a similar result holds for (1.3) with a general $J \geq 0$ and for (1.4). In [6, p. 251], the authors speculated that (1.6) admits a soliton connecting consecutive stable roots of any balanced bistable f, at least for small $\varepsilon > 0$. We think that one cannot expect much more beyond Theorem 2.1, and a proof of the aforementioned Hypothesis 1 in [5] would disprove this conjecture. Some methods were introduced for showing the nonexistence of monotonic heteroclinic solutions of a third order equation bearing some similarity to (1.7) [18, 8, 2, 9], however, they do not seem to be easily applicable here.

In [15] the author discovered that $u(y) = \tanh(ky)$ is a solution of

$$c^2 u''(y) = d(u(y+1) - 2u(y) + u(y-1)) - f(u(y)),$$

where

$$W(u) = (c^2k^2 - d)u^2 - \frac{c^2k^2}{2}u^4 - \frac{d}{\sinh^2 k}\ln(\cosh^2 k - u^2\sinh^2 k). \tag{1.9}$$

In Section 2, we derive (1.8) and discuss its consequences, e.g., Theorem 2.2, in which we construct a W with any number of arbitrarily spaced wells, admitting solitons connecting the outermost ones. This generalizes (1.9) in a nontrivial way, since W in (1.9) has only three equal depth wells for an appropriate choice of parameters. In Section 3, we consider (1.7) without the term $\varepsilon^2(f(u))''$, but with $f(u) = \sin u$. We find an exact 4π-kink (Theorem 3.1) and use the analytic continuation method from [2] to show that such a simpler equation admits no 2π-kinks (Theorem 3.2).

2. The main result

Let f be a balanced bistable nonlinearity. To be more precise, we assume that (2.1)

$$f \in C^1,\ f(\pm a) = f(0) = 0,\ f|_{(-a,0)} > 0,\ f|_{(0,a)} < 0,\ f'(\pm a) > 0,\ \int_{-a}^{a} f = 0.$$

We reach (1.8) as follows. Let $\lambda_c = \frac{1-c^2}{c^2}$ and $f_c = \frac{1}{c^2}f$. Then (1.7) becomes

$$\varepsilon^2 u^{(iv)} + \lambda_c u'' - f_c(u) + \varepsilon^2 f_c(u)'' = 0 \tag{2.2}$$

The reduction to a second order equation that follows is similar in spirit to that in [11] applied to the equation $-\gamma u^{(iv)} + u'' - f(u) = 0$, though the calculation here is trickier and the end result is different than the one in [11]. First we find the first integral of (2.2). Multipling (2.2) by u' and integrating from $-\infty$ to y leads to

$$\varepsilon^2 u'''u' - \frac{1}{2}\varepsilon^2 u''^2 + \frac{1}{2}\lambda_c u'^2 - W_c(u) + \varepsilon^2\int_{-\infty}^{y}(f_c(u))''u' = 0, \tag{2.3}$$

where $W_c = \frac{1}{c^2}W$. The integral term in (2.3) is handled in the following way. First we integrate twice by parts:

$$\begin{aligned}\varepsilon^2\int_{-\infty}^{y}(f_c(u))''u' &= \varepsilon^2 f_c'(u)u'^2 - \varepsilon^2\int_{-\infty}^{y}(f_c(u))'u'' \\ &= \varepsilon^2 f_c'(u)u'^2 - \varepsilon^2 f_c(u)u'' + \varepsilon^2\int_{-\infty}^{y} f_c(u)u'''.\end{aligned} \tag{2.4}$$

Then we integrate (2.2) and substitute

$$\varepsilon^2 u''' = \int_{-\infty}^{y} f_c(u) - \lambda_c u' - \varepsilon^2(f_c(u))'$$

into the integral term in (2.4):

$$\varepsilon^2\int_{-\infty}^{y} f_c(u)u''' = \frac{1}{2}[\varepsilon^2 u''' + \lambda_c u' + \varepsilon^2(f_c(u))']^2 - \lambda W_c(u) - \frac{1}{2}\varepsilon^2 f_c(u)^2. \tag{2.5}$$

Substituting (2.5) into (2.4), and (2.4) into (2.3), we obtain the first integral of (2.2):

$$\varepsilon^2 u''' u' - \frac{1}{2}\varepsilon^2 u''^2 + \frac{1}{2}\lambda_c u'^2 - W_c(u) + \varepsilon^2 f_c'(u) u'^2 - \varepsilon^2 f_c(u) u''$$

$$+\frac{1}{2}[\varepsilon^2 u''' + \lambda_c u' + \varepsilon^2 (f_c(u))']^2 - \lambda_c W_c(u) - \frac{1}{2}\varepsilon^2 f_c(u)^2 = 0. \tag{2.6}$$

We consider only solutions u such that $u' > 0$. Let $s = u(y)$ and $y(s) = u^{-1}(s)$. We reduce the order in (2.6) with the substitution $v(s) = [u'(y(s))]^2$. Note that $v'(s) = 2u''(y(s))$ and $v''(s) = \frac{2u'''}{u'}$. We get:

$$\frac{1}{2}\varepsilon^2 v'' v - \frac{1}{8}\varepsilon^2 v'^2 + \frac{1}{2}\lambda_c v + \varepsilon^2 f_c'(s) v - \frac{1}{2}\varepsilon^2 f_c(s) v'$$

$$+\frac{1}{2}[\frac{1}{2}\varepsilon^2 v''\sqrt{v} + \lambda_c \sqrt{v} + \varepsilon^2 f_c'(s)\sqrt{v}]^2 - \frac{1}{2}\varepsilon^2 f_c(s)^2 - (1+\lambda_c) W_c(s) = 0.$$

It is probably remarkable that this equation can be simplified with the substitution $z = \frac{1}{2}v + W_c(s)$, to give

$$-\frac{1}{2}\varepsilon^2 z'^2 + (z - W_c(s))(1 + \lambda_c + \varepsilon^2 z'')^2 - (1+\lambda_c) z = 0,$$

or,

$$-\frac{1}{2}\varepsilon^2 c^2 z'^2 + (c^2 z - W(s))(c^{-2} + \varepsilon^2 z'')^2 - z = 0, \tag{2.7}$$

which decouples the nonlinearity from the solution and yields (1.8).

A positive solution of (2.7) with the boundary conditions $z(\pm a) = z'(\pm a) = 0$ yields a solution of (1.7) with $u(\pm\infty) = \pm a$, if in addition

$$y(u) = \int_0^u y'(s) ds = \int_0^u \frac{ds}{\sqrt{v(s)}} \to \pm\infty \text{ as } u \to \pm a. \tag{2.8}$$

Let us assume that $\lim_{s\to a} z''(s)$ exists. Denote it by L. As in [11], after dividing both sides of (2.7) by z and passing to the limit, we get

$$\Big(c^2 - \lim_{s\to a}\frac{W(s)}{z(s)}\Big)(c^{-2} + \varepsilon^2 L)^2 = 1 + \frac{1}{2}\varepsilon^2 c^2 \lim_{s\to a}\frac{z'^2(s)}{z(s)}.$$

After using l'Hôpital's rule we get

$$\Big(1 - \frac{f'(a)}{Lc^2}\Big)(c^{-2} + \varepsilon^2 L) = 1, \tag{2.9}$$

or

$$\varepsilon^2 c^2 L^2 + (1 - c^2 - \varepsilon^2 f'(a)) L - c^{-2} f'(a) = 0,$$

which has the solutions

$$L_\pm = \frac{1}{2\varepsilon^2 c^2}\Big[-(1 - c^2 - \varepsilon^2 f'(a)) \pm \sqrt{(1 - c^2 - \varepsilon^2 f'(a))^2 + 4\varepsilon^2 f'(a)}\Big].$$

After squaring, $\frac{1}{2}v''(a) = L_+ - \frac{f'(a)}{c^2} > 0$ is equivalent to $1 > 1 - c^2$, hence for some $K > 0$, $v(s) \sim K(a-s)^2$ as $s \to a$. As a similar argument applies at $s \to -a$, we see that (2.8) is satisfied.

As an application of (1.8), we get the following results.

Theorem 2.1. *Let f_0 satisfy (2.1). For any $0 < c^2 < 1$, there is an $\varepsilon(c) > 0$, such that for $0 < \varepsilon < \varepsilon(c)$ there are solution pairs $(u_\varepsilon, f_\varepsilon)$ of (1.7), in the sense that u_ε, such that $u'_\varepsilon > 0$ and $u_\varepsilon(\pm\infty) = \pm a$, satisfies (1.7) with f_ε. Moreover, $(u_\varepsilon, f_\varepsilon) \to (u_0, f_0)$ as $\varepsilon \to 0$, where u_0 satisfies $(1-c^2)u''_0 - f_0(u_0) = 0$.*

Proof. Let $z_0(s) = \frac{1}{2}[u'_0(u_0^{-1}(s))]^2 + \frac{W_0(s)}{c^2}$. For $0 < \varepsilon < \varepsilon(c) = \sqrt{-\frac{1}{c^2 \min z''_0(t)}}$, we can define $W_\varepsilon = c^2 z_0 - \frac{z_0 + \frac{1}{2}\varepsilon^2 c^2 {z'_0}^2}{(\frac{1}{c^2} + \varepsilon^2 z''_0)^2}$ and $\frac{1}{2}v_\varepsilon = z_0 - \frac{W_\varepsilon}{c^2}$. Let $L_0 = \lim_{s\to a} z''_0(s)$. From (2.9), $f'_\varepsilon(a) = L_0 \frac{1-c^2+\varepsilon^2 L_0}{c^{-2}+\varepsilon^2 L_0} > 0$, thus $v''_\varepsilon(a) > 0$ and u_ε defined by $y(u_\varepsilon) = \int_0^{u_\varepsilon} \frac{ds}{\sqrt{v_\varepsilon(s)}}$ is a solution corresponding to $f_\varepsilon = W'_\varepsilon$. □

Remark. Recall that if f_0 is multistable, e.g., $f_0(u) = \sin u$, then $(1-c^2)u''_0 - f_0(u_0) = 0$ has only solitons connecting nearest stable zeroes of f_0. However, this is not the case for (1.7), as was e.g., determined numerically in [5]. Indeed, let z_0 correspond to $f_0(u) = u^3 - u$, i.e., $z_0(t) = \frac{(t^2-1)^2}{4c^2(1-c^2)}$. It can be verified that for $\varepsilon_t(c) = \sqrt{(1-c^2)(1-|c|)}$, $W_{\varepsilon_t(c)}$ is a triple-well function with wells of equal depth, i.e., $W_{\varepsilon_t(c)}(\pm 1) = W_{\varepsilon_t(c)}(0) = 0$. For $\varepsilon_t(c) < \varepsilon < \varepsilon(c)$, $W_\varepsilon(0) < 0$. This example is similar to (1.9) [15]. However, (1.8) enables us to go further.

Theorem 2.2. *Let $a_1, \ldots, a_n$ be an increasing sequence. For any $0 < c^2 < 1$, there exists a multi-well potential $\tilde{W}$, such that $\tilde{W}(a_k) = 0$, $k = 1, \ldots, n$, $\tilde{W} > 0$ elsewhere, and a soliton solution $\tilde{u}' > 0$ of (1.7) with $\tilde{f} = \tilde{W}'$, such that $\tilde{u}(-\infty) = a_1$, $\tilde{u}(+\infty) = a_n$.*

Proof. Let $\bar{z}$ have zeroes at $\pm m = \min\{a_2 - a_1, a_n - a_{n-1}\}$ and $\bar{\varepsilon}_t$ be the value corresponding to $W_{\bar{\varepsilon}_t}$ being a triple well function, as in the above Remark.

We paste and glue. Cover $a_2, \ldots, a_{n-1}$ with disjoint closed intervals $I_2, \ldots, I_{n-1}$, each of the same length less than $2m$. On each $I_k = [i_k, j_k]$, $k = 2, \ldots, n-1$, let $\tilde{z}(t) = \bar{z}(t - i_k - \frac{|I_k|}{2})$. Between those intervals extend $\tilde{z}$ smoothly and sufficiently close to a constant. Without loss of generality, let $[a_1, a_2]$ be shorter than $[a_{n-1}, a_n]$. On $[a_1, i_2]$ let $\tilde{z}(t) = \bar{z}(t - a_1 - m)$, on $[a_n - (i_2 - a_1), a_n]$ let $\tilde{z}(t) = \bar{z}(t - a_n + m)$. On $[j_{n-1}, a_n - (i_2 - a)]$ extend $\tilde{z}$ smoothly and sufficiently close to a constant.

Then $\tilde{W}$ defined by (1.8) with $z = \tilde{z}$ and $\varepsilon = \bar{\varepsilon}_t$ has the required properties. The soliton $\tilde{u}$ is obtained as in the Proof of Theorem 2.1. □

3. An asymptotic limit equation with sine

In [5] the authors suggested considering solutions of (1.7) of the form $U(y) = u(\sqrt{\varepsilon}y)$. U is then a solution of

$$c^2U^{(iv)} + \frac{1-c^2}{\varepsilon}U'' - f(U) + \varepsilon f(U)'' = 0. \tag{3.1}$$

Assuming $c^2 \to 1$, $\varepsilon \to 0$ and $\frac{1-c^2}{\varepsilon} \to \lambda > 0$, we obtain the simpler

$$U^{(iv)} + \lambda U'' - f(U) = 0. \tag{3.2}$$

First we present a similar result as the one for (1.5) in [1].

Theorem 3.1. *For* $\lambda = \frac{\sqrt{3}}{32}$

$$U(y) = 8\arctan\big(\exp(\frac{\sqrt{2}}{\sqrt[4]{3}}y)\big) \tag{3.3}$$

is an exact 4π*-kink solution of (3.2) with* $f(U) = \sin U$.

Remark. In [5, p.409], the authors' numerical results give the first such 4π-kink at $\lambda \approx 1.155$.

Proof of Theorem 3.1. We show how we found (3.3). Consider only solutions such that $U' > 0$ and let $v(s) = [U'(U^{-1}(s))]^2$, as in Section 2. Using the first integral of (3.2), we get

$$vv'' - \frac{v'^2}{4} + \lambda v - 2W(t) = 0. \tag{3.4}$$

We try if $v(s) = 1 - \cos\frac{s}{2}$ is an exact solution. For $\lambda = \frac{1}{4}$ it indeed satisfies (3.4) with $W(s) = \frac{3}{64}(1-\cos s)$, thus $\bar{U}(y) = 8\arctan(\exp(\frac{\sqrt{2}}{4}y))$ determined from

$$y(\bar{U}) = \int_{2\pi}^{\bar{U}} \frac{dt}{\sqrt{v(s)}} = \int_{2\pi}^{\bar{U}} \frac{ds}{\sqrt{2}\sin\frac{s}{4}} = \frac{4}{\sqrt{2}}\ln\tan(\frac{\bar{U}}{8})$$

satisfies

$$\bar{U}^{(iv)} + \frac{1}{4}\bar{U}'' - \frac{3}{64}\sin\bar{U} = 0.$$

Therefore $U(y) = \bar{U}(\frac{y}{\sqrt[4]{\frac{3}{64}}})$ satisfies (3.2) for $\lambda = \frac{\sqrt{3}}{32}$ and is given by (3.3). $\square$

However, there are no 2π-kinks for (3.2) with $f(U) = \sin U$. Since we will be working in the complex plane, it is convenient to switch the notation, so that $U \equiv U(x)$.

Theorem 3.2. *Let* $\lambda > 0$ *and* $f(U) = \sin U$. *There exist no solutions* U *of (3.2), such that* $U(-\infty) = 0$, $U(+\infty) = 2\pi$.

Proof. To simplify the notation, let $w(x) = -\pi + U(x)$. Then w satisfies

$$w^{(iv)} + \lambda w'' + \sin w = 0. \tag{3.5}$$

In Lemmas 3.3 and 3.4 below we show that $w' > 0$ and w is odd. With this at hand, we can adapt the analytic continuation approach method in [2], where the authors established a similar result for the equation $\varepsilon w''' + w' - \cos w = 0$, see also [3, 10] for extensions to some equations with polynomial nonlinearities. It is reminiscent of Painlevé transcendents and some properties of the inverse scattering theory. In this context, the inspiration might had also been drawn from some beyond all orders asymptotics results, e.g., [16, 14].

We argue by contradiction. First the solution w is analytically continued to $w(z)$, $z \in C$, in such a way that it retains some properties of $w_0(z) = -\pi + 4\arctan\exp\sqrt{\lambda}z$, which solves $\lambda w_{xx} + \sin w = 0$. Then, what we think is the main idea of the method, is that these properties are incompatible with the structure of (3.5) near the singularity of w. We will also see the limitation of this approach, namely, that at this stage there is compatibility with the structure of the complete nonlocal equation (1.7) with $f(u) = -\sin u$.

Recall that $\arctan z = \frac{1}{2i}\log\frac{1+iz}{1-iz}$. Using this representation, if we define log to have a branch cut along the positive real axis, then w_0 has branch cuts along the lines $z = x + i\sqrt{\lambda}(\frac{\pi}{2} + k\pi)$, $x < 0$, $k \in Z$. Moreover, if $w_0(z) = p_0(x,y) + iq_0(x,y)$, then $p_0 = \pi$ for $z = x + i\sqrt{\lambda}\frac{\pi}{2}$, $x > 0$, $p_0 = 0$ for $z = iy$, $0 \le y < \sqrt{\lambda}\frac{\pi}{2}$, $p_0 = 2\pi$ for $z = iy$, $\sqrt{\lambda}\frac{\pi}{2} < y \le \sqrt{\lambda}\pi$ and $p_0(x,y) = 2\pi - p_0(x, \sqrt{\lambda}\pi - y)$, $q_0(x,y) = q_0(x, \sqrt{\lambda}\pi - y)$ for $x > 0$, $0 < y \le \sqrt{\lambda}\frac{\pi}{2}$.

Let m_1 denote the positive root of $m^4 + \lambda m^2 - 1 = 0$. Below we show that w can be extended to $w(z)$, where $z = x + iy$, $x \ge 0$, $0 \le y \le \frac{\pi}{m_1}$, $y \ne \frac{1}{2}\frac{\pi}{m_1}$, which is a solution of $w_{xxxx} + \lambda w_{xx} + \sin w = 0$, in such a way that if $w(z) = p(x,y) + iq(x,y)$, then

$$
\begin{aligned}
&p(x, \tfrac{1}{2}\tfrac{\pi}{m_1}) = \pi,\ x > 0,\\
&q(x,0) = 0,\ x > 0,\\
&p(0,y) = 0,\ 0 \le y < \tfrac{1}{2}\tfrac{\pi}{m_1},\\
&p(0,y) = 2\pi,\ \tfrac{1}{2}\tfrac{\pi}{m_1} < y \le \tfrac{\pi}{m_1},\\
&p(x,y) = 2\pi - p(x, \tfrac{\pi}{m_1} - y) \text{ and}\\
&q(x,y) = q(x, \tfrac{\pi}{m_1} - y) \text{ for } x > 0,\ 0 < y \le \tfrac{1}{2}\tfrac{\pi}{m_1}.
\end{aligned}
\tag{3.6}
$$

At $z = i\frac{1}{2}\frac{\pi}{m_1}$, w has a singularity. We extend w as a solution of the equation to the left hand strip:

$$
\begin{aligned}
&p(-x,y) = -p(x,y),\ 0 \le y < \tfrac{1}{2}\tfrac{\pi}{m_1},\\
&p(-x,y) = 4\pi - p(x,y),\ \tfrac{1}{2}\tfrac{\pi}{m_1} < y \le \tfrac{\pi}{m_1},\\
&q(-x,y) = q(x,y),\ 0 \le y \le \tfrac{\pi}{m_1}.
\end{aligned}
$$

Thus w is analytic in the whole strip $S = \{(x,y) : 0 \le y \le \frac{\pi}{m_1}\}$, with the exception of the half line $z = x + i\frac{1}{2}\frac{\pi}{m_1}$, $x \le 0$, across which p is discontinuous:

$$\lim_{y \to \frac{1}{2}\frac{\pi}{m_1}+} p(x,y) - \lim_{y \to \frac{1}{2}\frac{\pi}{m_1}-} p(x,y) = 4\pi, \ x < 0.$$

Let

$$h(z) = w(z) + 2i \log(z - i\tfrac{1}{2}\tfrac{\pi}{m_1}), \tag{3.7}$$

where this log has a branch cut across the negative real axis. Since h is continuous across the half line $z = x + i\frac{1}{2}\frac{\pi}{m_1}$, $x \le 0$, it is also analytic in S, with the exception of the point $(0, \frac{1}{2}\frac{\pi}{m_1})$. Let $h = p_1 + iq_1$. Since $p_1 = p - 2\arg(z - i\frac{1}{2}\frac{\pi}{m_1})$ and p is bounded, p_1 is also bounded. From Big Picard Theorem, h cannot have an essential singularity at $(0, \frac{1}{2}\frac{\pi}{m_1})$. An elementary calculation also shows that if p_1 is bounded, the singularity of h cannot be a pole. Thus h is analytic. Substituting (3.7) into (3.5), we obtain a contradiction, since w_{xx} and $\sin w$ are both of order $O((z - i\frac{1}{2}\frac{\pi}{m_1})^{-2})$, while w_{xxxx} is of order $O((z - i\frac{1}{2}\frac{\pi}{m_1})^{-4})$.

Remark. There is no such contradiction if we consider (1.7) in place of (3.5), as $(\sin w)_{xx}$ is also of order $O((z - i\frac{1}{2}\frac{\pi}{m_1})^{-4})$. $\square$

To complete the proof, we prove the aforementioned lemmas and construct the analytic continuation of $w(x)$ with the needed properties.

Lemma 3.3. *$w' > 0$.*

Proof. First we need to show that $w', w'', w''' \to 0$ as $x \to \pm\infty$. (3.5) can be written as

$$w'' = (\lambda + 1)J * w - (\lambda + 1)w + J * \sin w, \tag{3.8}$$

where $J(x) = \frac{1}{2}e^{-|x|}$. We used $\lim_{x \to \pm\infty} e^{-|x|}w'''(x) = 0$, which is not hard to get from (3.5). From (3.8) and Lebesgue Dominated Convergence Theorem, $w'' \to 0$ as $x \to \pm\infty$. Differentiating (3.8) we get

$$w''' = (\lambda + 1)J' * w - (\lambda + 1)w' + J' * \sin w, \tag{3.9}$$

which can be written as

$$w' = -(\lambda + 2)J' * w - (\lambda + 1)J * J' * w - J * J' * \sin w,$$

thus $w' \to 0$ as $x \to \pm\infty$ and $w''' \to 0$ as $x \to \pm\infty$ from (3.9).

Now we can use the first integral of (3.5)

$$w'''w' - \frac{1}{2}w''^2 + \frac{\lambda}{2}w'^2 = 1 + \cos w,$$

from which we see that we can have $w' = 0$ only at points at which $w'' = 0$ and $1 + \cos w = 0$. If $w' = 0$ at such a point, then to satisfy $w(\pm\infty) = \pm\pi$, w must be

at a local maximum or minimum at this or another such a point. From Taylor's expansion to fourth order, also $w''' = 0$ at that point, but then from uniqueness of solutions of the initial value problem, w is a constant solution, thus reaching a contradiction. $\square$

If we write (3.5) as a system of first order equations, we see that the critical point $(\pi, 0, 0, 0)$ is not hyperbolic. This is in contrast to the critical point $(\frac{\pi}{2}, 0, 0)$ of the third order equation in [2], which is hyperbolic, therefore some arguments taken from [2] need a bit of care here. Let m_2 denote the negative root of $m^4 + \lambda m^2 - 1 = 0$ and m_3, m_4 its purely complex ones. Note that $m_1 + m_2 + m_3 + m_4 = 0$.

Lemma 3.4. *$\pi - w(x) = O(e^{-m_1 x})$ as $x \to \infty$ and w is odd.*

Proof. If we could show that $\lim_{s\to\pi} v''(s)$ and $\lim_{s\to\pi} v'''(s)$ exist in the representation (3.4), then we would calculate these limits as in (2.9), in particular getting $\lim_{s\to\pi} v''(s) = 2m_1^2$, and then get the asymptotics from $x(w) - \bar{x} = \int_{w(\bar{x})}^{w} \frac{ds}{\sqrt{v(s)}}$, where $\bar{x}$ is sufficiently large. Such an argument was used in [11]. However, since it is not easy to show that the limits exist, we use the more robust method from [7].

Let $\bar{w} = \pi - w$, $w_1 = \int_x^\infty \bar{w}$, $w_2 = \int_x^\infty w_1$. Note that $\bar{w}$ and w_1 are integrable from (3.5) and $w', w'', w''' \to 0$ as $x \to \infty$. For small $\delta > 0$ and large x for which $\bar{w} < \delta$ we have

$$\bar{w}'' + \lambda \bar{w} \geq (1 + O(\delta)) w_2.$$

Let A be the set on which $\bar{w} \geq \bar{w}''$, B the set on which $\bar{w} < \bar{w}''$. Since for any $x_1 > 0$ we have $\int_x^\infty \bar{w} \geq \int_x^{x+x_1} \bar{w} \geq x_1 \bar{w}(x + x_1)$, on A we get

$$\frac{1+\lambda}{1+O(\delta)} \bar{w} \geq \int_x^\infty x_1 \bar{w}(s + x_1) ds \geq x_1 \int_x^{x+x_1} \bar{w}(s + x_1) ds \geq x_1^2 \bar{w}(x + 2x_1).$$

In a similar manner, on B we get

$$\frac{1+\lambda}{1+O(\delta)} \bar{w} \geq x_1^4 \bar{w}(x + 4x_1).$$

Thus for all large $\bar{x}$ there is a $k < 1$ such that $\bar{w}(x+\bar{x}) \leq k\bar{w}(x)$. Let $h(x) = \bar{w}(x)e^{\gamma x}$, where $\gamma = \frac{1}{\bar{x}} \ln \frac{1}{k}$. Then

$$h(x + \bar{x}) = \bar{w}(x + \bar{x}) e^{\gamma x} e^{\gamma \bar{x}} \leq \bar{w}(x) e^{\gamma x} = h(x),$$

and thus h is bounded and $\pi - w(x) = O(e^{-\gamma x})$ as $x \to \infty$. Thus for $-\gamma < \mathrm{Re}\ \xi < 0$ we can define the two-sided Laplace transform of $\bar{w}$ by $W(\xi) = \int_R e^{-\xi x} \bar{w}(x) dx$, and it satisfies

$$(\xi^4 + \lambda \xi^2 - 1) W(\xi) = \int_R e^{-\xi x} r(\bar{w}) dx, \tag{3.10}$$

where $r(\bar{w}) = -\frac{1}{6}\bar{w}^3 + \ldots$. Since the right side in (3.10) is defined for $-3\gamma < \mathrm{Re}\ \xi < 0$, by bootstrap W is analytic in the strip $-m_1 < \mathrm{Re}\ \xi < 0$. Since $\bar{w}$ is a positive

decreasing function and $\int_0^\infty e^{-\xi x}\bar{w}(x)dx = \frac{H(\xi)}{\xi+m_1}$, where H is analytic in the strip $-m_1 \le \text{Re}\ \xi < 0$, from Ikehara's Theorem we conclude that $\pi - w(x) = O(e^{-m_1 x})$ [7].

Using the method of variation of parameters, $\bar{w}$ satisfies the integral equation

$$\begin{aligned}\bar{w}(x) \quad &= \quad c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + c_4 e^{m_4 x} \\ &\quad + a_1 e^{m_1 x}\int_x^\infty e^{(m_2+m_3+m_4)s} r(\bar{w}(s))ds \\ &\quad + a_2 e^{m_2 x}\int_x^\infty e^{(m_1+m_3+m_4)s} r(\bar{w}(s))ds \\ &\quad + a_3 e^{m_3 x}\int_x^\infty e^{(m_1+m_2+m_4)s} r(\bar{w}(s))ds \\ &\quad + a_4 e^{m_4 x}\int_x^\infty e^{(m_1+m_2+m_3)s} r(\bar{w}(s))ds,\end{aligned}$$

where a_i, $i = 1, \dots, 4$, can be calculated expicitly from the method and necessarily $c_1 = c_3 = c_4 = 0$.

Let $c = c_1$, $\bar{w}_0(z) = ce^{-m_1 z}$, and the sequence $\bar{w}_n(z)$ be defined by

$$\begin{aligned}\bar{w}_{n+1}(z) = ce^{-m_1 z} + a_1 e^{m_1 z}\int_z^\infty e^{-m_1 s} r(\bar{w}_n(s))ds + a_2 e^{-m_1 z}\int_z^\infty e^{m_1 s} r(\bar{w}_n(s))ds \\ + a_3 e^{m_3 z}\int_z^\infty e^{-m_3 s} r(\bar{w}_n(s))ds + a_4 e^{-m_3 z}\int_z^\infty e^{m_3 s} r(\bar{w}_n(s))ds,\end{aligned}$$

where the integration paths are on the horizontal line $z = x+iy$, with y fixed. There exists M such that for Re $z \ge M$ and $0 \le \text{Im}\ z \le \frac{\pi}{m_1}$ we have $|\bar{w}_n(z)| \le 2ce^{-m_1 x}$ for all n and $\bar{w}_n(z)$ converges uniformly to a unique solution $\bar{w}(z)$. Since each $\bar{w}_n$ is analytic and the convergence is uniform, $\bar{w}$ is also analytic. Using Picard iterations in which the integrations are on bounded horizontal segments, $\bar{w}$ is then analytically extended to the maximal strip of existence $\{(x, y) : x > x_s, 0 \le y \le \frac{\pi}{m_1}\}$.

To show that $\bar{w}(x)$ is odd, first note that $\tilde{w}(x) = -\bar{w}(-x)$ is also a solution of (3.5). Let c_0 correspond to $\tilde{w}(0) = 0$. Since the solution of the Picard iteration with $c = c_0$ is unique, $\tilde{w} = \bar{w}$ and $\bar{w}$ is odd. □

To show (3.6), we first establish various monotonicity properties of p and q. Let $0 \le y \le \frac{1}{2}\frac{\pi}{m_1}$. Since $\bar{w}(z) = e^{-m_1 z}[c + o(1)]$, $p < \pi$ and $q > 0$ for x large enough. Also, from

$$\begin{aligned}&p_x = m_1 e^{-m_1 x}\cos m_1 y\ (1 + o(1)),\\ &q_x = -m_1 e^{-m_1 x}\sin m_1 y\ (1 + o(1)),\\ &p_{xx} = -m_1^2 e^{-m_1 x}\cos m_1 y\ (1 + o(1)),\\ &q_{xx} = m_1^2 e^{-m_1 x}\sin m_1 y\ (1 + o(1)),\end{aligned}$$

we conclude that

$$p_x > 0,\ q_x < 0,\ p_{xx} < 0,\ q_{xx} > 0 \text{ for } x \text{ large enough.}$$

Note that $w = p + iq$ satisfies the system

$$\left\{\begin{array}{l} p^{(iv)} + \lambda p'' + \sin p \ \cosh q = 0, \\ q^{(iv)} + \lambda q'' + \cos p \ \sinh q = 0, \end{array}\right. \tag{3.11}$$

so that

$$\begin{array}{l} p_x = \frac{1}{\sqrt{\lambda}} \int_x^\infty (1 - \cos \sqrt{\lambda}(x-s)) \ \sin p(s,y) \ \cosh q(s,y) \ ds, \\ p_{xx} = \frac{1}{\sqrt{\lambda}} \int_x^\infty (1 - \cos \sqrt{\lambda}(x-s)) \ (\cos p(s,y) \ p_s(s,y) \ \cosh q(s,y), \\ \qquad + \sin p(s,y) \ \sinh q(s,y) \ q_s(s,y)) \ ds, \\ q_x = \frac{1}{\sqrt{\lambda}} \int_x^\infty (1 - \cos \sqrt{\lambda}(x-s)) \ \cos p(s,y) \ \sinh q(s,y) \ ds, \end{array} \tag{3.12}$$

Since $|w(z)| \le 2ce^{-m_1 x}$ uniformly in y, from (3.12) we see that $p_x > 0$, $q_x < 0$ and $p_{xx} < 0$ hold on $S = \{(x,y) : x \ge \max\{M, p(\cdot,0)^{-1}(\frac{\pi}{2})\}, 0 \le y \le \frac{1}{2}\frac{\pi}{m_1}\}$.

Let $\pi - \bar{w}_n = p_n + iq_n$. Since all p_n and q_n satisfy (3.6), with p_n in place of p and q_n in place of q, (3.6) holds on S also for their limits p and q.

From (3.12) we see that x_s is the value at which $q(x,y)$ becomes infinite as $x \to x_s+$. From $p(x, \frac{1}{2}\frac{\pi}{m_1}) = \pi$ and the last equation in (3.12), we see that as long as $p_y, q_y \ge 0$ in $\{(x,y) : x > x_s, 0 \le y \le \frac{1}{2}\frac{\pi}{m_1})$, this singularity will be on the line $y = \frac{1}{2}\frac{\pi}{m_1}$.

Let $S_s = \{(x,y) : x > \max\{0, x_s\}, 0 < y < \frac{1}{2}\frac{\pi}{m_1}\}$. Since $q_y = p_x$, from the first equation in (3.12) we see that the first value x_f at which p_y or q_y is 0 in S_s can be only such that $p_y = 0$. However, from (3.11) we get

$$- p_{yyy} + p_y = \int_0^y \sin p \ \cosh q, \tag{3.13}$$

thus since 0 would be a minimal value for p_y in $I_f = \{(x,y) : x = x_f, 0 < y < \frac{1}{2}\frac{\pi}{m_1}\}$ and the right side of (3.13) is positive, we get a contradiction. Also, since p_y is harmonic and positive in S_s, from the maximum principle we cannot have $p_y = p_{yy} = 0$ on $\{(x,0) : x > \max\{0, x_s\}\}$. Thus $p_y, q_y > 0$ in S_s and $p_{xx} < 0$ on $\{(x,0) : x > \max\{0, x_s\}\}$. In a similar way we obtain that $q_{xx} > 0$ on $\{(x, \frac{1}{2}\frac{\pi}{m_1}) : x > \max\{0, x_s\}\}$ and $p_{yy} > 0$ in S_s.

Let $x_s \ge 0$ and $I_s = \{(x,y) : x = x_s, 0 < y < \frac{1}{2}\frac{\pi}{m_1}\}$. In a similar setting, in [10] the authors studied the analytic continuation of w on I_s, using Harnack's inequality and the polynomial form of the nonlinearity. They reached a contradiction and avoided the eventual analytic continuation to the left-half plane. Such an argument seems unavailable for a transcendental function, so we argue as in [2], adding a few more details for the convenience of the reader.

Since $p(\cdot, y)$ is decreasing and bounded in S_s, let $p(x_s, y) = \lim_{x \to x_s+} p(x,y)$. From the last equation in (3.12), we see that there is no $y_s \in (0, \frac{1}{2}\frac{\pi}{m_1})$ for which $q_x(x, y_s) \to -\infty$ as $x \to x_s+$, as then we would have $p_y(x,y) = -q_x(x,y) \to \infty$ as $x \to x_s+$ for all $y \in (y_s, \frac{1}{2}\frac{\pi}{m_1})$, contradicting that p is bounded. Let $x_n \to x_s+$.

For any $\delta > 0$, $p_y(x_n, y)$ is a sequence of increasing and bounded functions on $[0, \frac{1}{2}\frac{\pi}{m_1} - \delta]$, so from Helly's Theorem there is a subsequence converging to an increasing function $h(y) = \lim_{x_n \to x_s+} p_y(x_n, y)$. Passing to the limit on both sides of $p(x_n, y) = p(x_n, 0) + \int_0^y p_s(x_n, s)ds$, we get $p_y(x_s, y) = h(y)$. Extending in this way to $y \in [0, \frac{1}{2}\frac{\pi}{m_1})$, we find that $p_y(x_s, y)$ is increasing.

Integrating the first equation in (3.11) twice, we get

$$-p_{xx}(x,0) + \lambda(\pi - p(x,0)) = \int_0^{\frac{1}{2}\frac{\pi}{m_1}} \int_0^y \sin p \ \cosh q.$$

Integrating the second equation in (3.11) twice, we get

$$q_{xx}(x, \tfrac{1}{2}\tfrac{\pi}{m_1}) + \lambda q(x, \tfrac{1}{2}\tfrac{\pi}{m_1}) - \frac{\lambda}{2}\frac{\pi}{m_1}(p_{xxx}(x,0) + p_x(x,0)) = \int_0^{\frac{1}{2}\frac{\pi}{m_1}} \int_0^y \cos p \ \sinh q.$$

If $p(x_s, 0) \not\equiv 0$, then there are some positive constants K_1, K_2, such that

$$\lim_{x \to x_s+} \int_0^{\frac{1}{2}\frac{\pi}{m_1}} \int_0^y \cos p \ \sinh q \le K_1 + \lim_{x \to x_s+} K_2 \int_0^{\frac{1}{2}\frac{\pi}{m_1}} \int_0^y \sin p \ \cosh q.$$

Since $q_{xx}(x, \frac{1}{2}\frac{\pi}{m_1}) > 0$ and $q(x, \frac{1}{2}\frac{\pi}{m_1}) \to \infty$ as $x \to x_s+$, we get a contradiction.

To reach (3.6), it is now enough to rule out the case $x_s < 0$. Since the solution of the initial value problem

$$\begin{cases} -p_{yyyy} + \lambda p_{yy} = \sin p \ \cosh q, \\ p(0,0) = p_y(0,0) = p_{yy}(0,0) = p_{yyy}(0,0) = 0, \end{cases}$$

is $p(0, y) \equiv 0$, it contradicts $p(0, \frac{1}{2}\frac{\pi}{m_1}) = \pi$. $\square$

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